Joint estimation of real squeezing and displacement

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We study the problem of joint estimation of real squeezing and amplitude of the radiation field, deriving the measurement that maximizes the probability density of detecting the true value of the unknown parameters. More generally, we provide a solution for the problem of estimating the unknown unitary action of a nonunimodular group in the maximum likelihood approach. Remarkably, in this case the optimal measurements do not coincide with the so called square-root measurements. In the case of squeezing and displacement we analyze in detail the sensitivity of estimation for coherent states and displaced squeezed states, deriving the asymptotic relation between the uncertainties in the joint estimation and the corresponding uncertainties in the optimal separate measurements of squeezing and displacement. A two-mode setup is also analyzed, showing how entanglement between optical modes can be used to approximate perfect estimation.

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I. INTRODUCTION

Squeezing and displacement are the basic operations of continuous variables quantum information [1], and are easily performed, the former by parametric amplifiers, the latter by lasers and linear optics. Squeezing in the two-mode setup is, for example, the tool to generate entanglement in the Braunstein-Kimble teleportation scheme [2]. The combined use of real squeezing and displacement allows one to encode efficiently classical information in quantum channels using homodyne detection at the receiver [3]. In a quantum communication scenario where a coherent signal is sent through a non-linear medium and undergoes an amplification process, the joint estimation of displacement and squeezing provides a twofold information: the amplitude modulation of the state and a property of the communication channel itself. This may be useful in a communication scheme designed to be robust to photon loss.

In this paper we consider the problem of jointly estimating real amplitude and real squeezing of the radiation field. In a tomographic setup, where a large number of equally prepared copies is available, the maximum likelihood method turns out to be very efficient in estimating the parameters that characterize the state of the quantum system [4, 5]. In this approach, first one fixes a set of single-copy measurements (typically homodyne measurements at some random phase) and then looks for the estimate that maximizes the probability (density) of producing the observed data. However, the tomographic approach is not suitable to the case when only a small number of copies is available, and one needs to use the limited resources at disposal more efficiently. It becomes then important to optimize not only the posterior processing of the experimental data, but also the choice of the measurement that is used to extract these data from the system. The most natural framework to deal with this situation is quantum estimation theory [6, 7], where the concept of positive operator valued measure (POVM) provides a tool to describe at the same time both measurement and data processing. The maximum likelihood approach in quantum estimation theory [8] then corresponds to seek the measurement that maximizes the probability (density) that the estimated value of the parameters coincides with the true value.

Joint estimation of squeezing and displacement is equivalent to infer an unknown transformation of a group—in the present case the affine group. This is an example of a frequent situation in quantum estimation, especially in communication problems, where a set of signal states is generated from a fixed input state by the action of a group. Consider, for example, the case of phase estimation for high-sensitivity interferometry and optimal clocks [9, 10, 11],

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the estimation of rotation for the optimal alignment of reference frames [12], and the estimation of displacement of the radiation field for the detection of a coherent signal in Gaussian noise [13]. In these cases, the symmetry of the problem provides a physical insight that allows one to simplify the search for efficient estimation strategies, and the concept of *covariant* measurement [7] becomes crucial for optimization. The use of maximum likelihood approach in the covariant setting has been shown to be particularly successful in obtaining explicitly the optimal measurements and in understanding the fundamental mechanism that leads to the ultimate sensitivity of quantum measurements [14, 15]. Group theoretical tools, such as equivalent representations [14] and multiplicity spaces [15], far from being abstract technicalities, are the main ingredients to achieve the ultimate quantum limits for sensitivity [12, 16]. Moreover, in a large number of situations, the maximization of the likelihood provides measurements that are optimal also according to a wide class of different figures of merit [16].

From a group theoretical point of view, the action of real squeezing and displacement on the wavefunctions provides a unitary representation of the affine group "$ax + b$" of dilations and translations on the real line. The structure of the affine group underlies the theory of wavelets [17], and has been recently used in the characterization of $SU(1, 1)$ coherent states [18, 19], and in the study of oscillators in a Morse potential [20].

The affine group is particularly interesting, since it is the paradigmatic example of a nonunimodular group, namely a group where the left-invariant Haar measure is different from the right-invariant one. This leads to orthogonality relations that—differently from the usual Schur lemmas—involve a positive unbounded operator, firstly introduced by Duflo, Moore, and Carey (DMC) in Ref. [21]. As we will see in this paper, the nonunimodularity of the group has some amazing consequences in the estimation problem. For example, the so-called square-root measurements [22], that are commonly considered in quantum communication and cryptography, do not coincide with the maximum-likelihood measurements, the latter providing a higher probability density of correct estimation. Another bizarre feature is that for maximum likelihood measurements, the most likely value in the probability distribution can be different from the true one. While for unimodular groups this feature never happens, for nonunimodular groups it is unavoidable, and a suitable choice of the input states is needed to reduce the discrepancy between the true value and the most likely one.

The paper is organized as follows. In order to set the optimal joint estimation of real squeezing and displacement in the general estimation method, first we derive in Sec. II the optimal measurement for the problem of estimating the unknown unitary action of a nonunimodular group. Then, as a special example, the general results will be used to optimize the joint estimation of squeezing and displacement in Sec. III. The efficiency of coherent states and displaced squeezed states is analyzed in detail, and the asymptotic relation between the uncertainties in the joint estimation and the uncertainties in the optimal separate measuremnts of squeezing and displacement is derived. The conclusions are summarized in Sec. IV. The explicit derivation of group average over the affine group is given in the Appendix.

II. MAXIMUM LIKELIHOOD ESTIMATION FOR A NONUNIMODULAR GROUP

A. Background and generalities

Suppose that a fixed input state ρ, corresponding to a density operator in the Hilbert space $\mathcal{H}$, is transformed by the unitary representation $\{U_g \mid g \in \mathbf{G}\}$ of the group $\mathbf{G}$, so that it generates the family of signal states

$$\mathcal{O} = \{U_g \rho U_g^\dagger \mid g \in \mathbf{G}\} \ . \tag{1}$$

The typical quantum estimation problem is then to find the measurement that gives the best estimate for the unknown transformation g according to some optimality criterion. Usually the criterion is given by a cost function $c(\hat{g}, g)$, which quantifies the cost of estimating $\hat{g}$ when the true value is g, and enjoys the invariance property $c(h\hat{g}, hg) = c(\hat{g}, g) \quad \forall h, \hat{g}, g \in \mathbf{G}$. For example, the maximum likelihood criterion corresponds to the delta cost-function $c(\hat{g}, g) = -\delta(g^{-1}\hat{g})$ (loosely speaking, there is an infinite gain if the estimate coincides with the true value, and no gain otherwise). Once a cost function is fixed, one can choose two possible approaches to optimization, namely the Bayes approach and the minimax. In the Bayes approach, one assumes some prior distribution of the unknown parameters, and then minimizes the average over the true values of the expected cost $c(g) = \int \mathrm{d}\,\hat{g}\; c(\hat{g}, g) p(\hat{g}|g)$, where $p(\hat{g}|g)$ is the conditional probability density of estimating $\hat{g}$ when the true value is g. In the minimax approach, one looks instead for the measurement that minimizes the supremum of the expected cost over all possible true values of the unknown parameters.

An important class of estimation strategies is given by the *covariant* measurements, that are described by POVMs of the form [7]

$$M(g) = U_g \xi U_g^\dagger \ , \tag{2}$$

where $\xi \geq 0$ is an operator satisfying the normalization condition

$$\int_{\mathbf{G}} \mathrm{d}_L\, g\ U_g \xi U_g^\dagger = \mathbb{1}\ , \tag{3}$$

$\mathrm{d}_L\, g$ denoting the left-invariant Haar measure on the group, namely $\mathrm{d}_L(hg) = \mathrm{d}_L\, g \quad \forall h, g \in \mathbf{G}$. Due to the symmetry of the set of states (1), covariant measurements play a fundamental role in the search of the optimal estimation. For compact groups, the following proposition holds:

Proposition 1 (Holevo [7]). *For compact groups, the search for the optimal measurement in the minimax approach can be restricted without loss of generality to the class of covariant measurements.*

Moreover, for compact groups the optimality of covariant measurements holds also in the Bayes approach [7], if the prior distribution is chosen to be the normalized Haar measure on the group, i.e. the measure $\mathrm{d}\, g$ such that $\mathrm{d}\, g = \mathrm{d}(hg) = \mathrm{d}(gh) \quad \forall g, h \in \mathbf{G}$ and $\int_{\mathbf{G}}\ \mathrm{d}\, g = 1$.

For non-compact groups, such as the affine group involved in the joint estimation of squeezing and displacement, the situation is more involved. Of course, in the Bayes approach it is no longer possible to choose the uniform Haar measure as prior distribution, since it is not normalizable. However, in the minimax the optimality of covariant measurements still holds, even though in this case the proof becomes rather technical [23]. In this paper we will adopt the minimax approach to deal with noncompact groups, and this will allow us to restrict the optimization to covariant POVMs.

B. Nonunimodular groups

As we will see in Section III, the action of real squeezing and displacement on one mode of the radiation field yields a representation of the affine group "$ax + b$" of dilations and translations on the real line. This group is clearly non-compact, and, moreover, it is nonunimodular, namely the left-invariant Haar measure $\mathrm{d}_L\, g$ (with $\mathrm{d}_L(hg) = \mathrm{d}_L\, g \quad \forall h, g \in \mathbf{G}$) does not coincide with the right-invariant one $\mathrm{d}_R\, g$ (with $\mathrm{d}_R(gh) = \mathrm{d}_R\, g \quad \forall h, g \in \mathbf{G}$). Therefore, to face the estimation problem with the affine group, we need some results about representation theory and orthogonality relations for nonunimodular groups (for an introduction to these topics, see for example [24]).

Let $\{U_g \mid g \in \mathbf{G}\}$ be a unitary representation of a locally compact group $\mathbf{G}$. In the following, we will make two assumptions on the representation $\{U_g\}$ that are tailored on the concrete problem of estimating real squeezing and displacement.

First assumption: discrete Clebsch-Gordan series. We require the representation $\{U_g\}$ to be a direct sum of irreducible representations (*irreps*), namely that its Clebsch-Gordan series is discrete. In this case, there is a decomposition of the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{\mu \in \mathsf{S}} \mathcal{H}_\mu \otimes \mathcal{M}_\mu\ , \tag{4}$$

such that

$$U_g = \bigoplus_{\mu \in \mathsf{S}} U_g^\mu \otimes \mathbb{1}_{\mathcal{M}_\mu}\ , \tag{5}$$

where $\{U_g^\mu\}$ is an irreducible representation acting on the Hilbert space $\mathcal{H}_\mu$ and $\mathbb{1}_{\mathcal{M}_\mu}$ is the identity in the space $\mathcal{M}_\mu$. The Hilbert spaces $\mathcal{H}_\mu$ and $\mathcal{M}_\mu$ are called *representation* and *multiplicity* spaces, respectively, and the index μ labels the inequivalent irreps that appear in the Clebsch-Gordan series S of the representation $\{U_g\}$.

Second assumption: square-summable irreps. We require each irreducible representation $\{U_g^\mu\}$ in Eq. (5) to be square-summable. This means that there is at least one non-zero vector $|\psi_\mu\rangle \in \mathcal{H}_\mu$ such that

$$\int_{\mathbf{G}} \mathrm{d}_L\, g\ |\langle \psi_\mu | U_g^\mu | \psi_\mu \rangle|^2 < \infty\ . \tag{6}$$

Vectors such that the above integral converges are called *admissible*. It is possible to show [24] that, if a representation is square-summable, then the set of admissible vectors is dense in the Hilbert space.

Let us consider now the group average of an operator A, defined as

$$\langle A \rangle_{\mathbf{G}} = \int_{\mathbf{G}} \mathrm{d}_L\, g\ U_g A U_g^\dagger\ . \tag{7}$$

In general, the average may not converge for any operator (for example, it diverges for $A = \mathbb{1}$). In analogy with admissible vectors, we say that A is an *admissible operator* if the group average in Eq. (7) converges in the weak operator sense. In such a case, one can prove that $\langle A\rangle_{\mathbf{G}}$ is given by

$$\langle A\rangle_{\mathbf{G}} = \bigoplus_{\mu\in\mathsf{S}} \mathbb{1}_{\mathcal{H}_\mu} \otimes \mathrm{Tr}_{\mathcal{H}_\mu}[D_\mu \otimes \mathbb{1}_{\mathcal{M}_\mu}\ A]\ , \tag{8}$$

where D_μ is a positive self-adjoint operator acting on the representation space $\mathcal{H}_\mu$. The operator D_μ has been firstly introduced by Duflo, Moore, and Carey (DMC) [21], and is the characteristic feature of nonunimodular groups. In fact, if the group $\mathbf{G}$ is unimodular—i.e. if the left- and right-invariant measures coincide—than the DMC operator is simply a multiple of the identity, and the formula (8) for the group average is equivalent to the ordinary Schur lemmas. Contrarily, if the group $\mathbf{G}$ is nonunimodular, the DMC operator is a positive unbounded operator, and its presence modifies the orthogonality relations dramatically, with remarkable consequences in the estimation of an unknown group transformation.

The admissibility of vectors and operators has a simple characterization in terms of the DMC operator. As regards vectors, the set of admissible vectors for the irrep $\{U_g^\mu\}$ can be characterized as the domain of $\sqrt{D_\mu}$. For unimodular groups, since D_μ is proportional to the identity, the set of admissible vectors is the whole representation space $\mathcal{H}_\mu$, while for nonunimodular groups the admissible vectors form a dense subset of $\mathcal{H}_\mu$. As regards operators, an operator A is admissible if and only if the partial traces $\mathrm{Tr}_{\mathcal{H}_\mu}[D_\mu \otimes \mathbb{1}_{\mathcal{M}_\mu} A]$ are not diverging. For example, if $|\psi_\mu\rangle \in \mathcal{H}_\mu$ is an admissible vector and O_μ is any operator acting on $\mathcal{M}_\mu$, then the operator $A_\mu = |\psi_\mu\rangle\langle\psi_\mu| \otimes O_\mu$ is admissible, and its group average is $\langle A_\mu\rangle_{\mathbf{G}} = \langle\psi_\mu|D_\mu|\psi_\mu\rangle\ \mathbb{1}_\mu \otimes O_\mu$.

C. Maximum likelihood measurements

In order to find the best estimate for the signal states (1) according to the maximum likelihood criterion in the minimax approach, we consider now a covariant measurement, described by a POVM $M(g)$ as in Eq. (2). The normalization condition (3) can be rewritten as

$$\int_{\mathbf{G}} \mathrm{d}_L\, g\ M(g) = \langle\xi\rangle_{\mathbf{G}} = \mathbb{1}\ . \tag{9}$$

Using Eq. (8) for evaluating the group average $\langle\xi\rangle_{\mathbf{G}}$, this condition becomes

$$\mathrm{Tr}_{\mathcal{H}_\mu}[D_\mu \otimes \mathbb{1}_{\mathcal{M}_\mu}\ \xi] = \mathbb{1}_{\mathcal{M}_\mu}\ . \tag{10}$$

According to the maximum likelihood approach, we need to find the covariant measurement that maximizes the probability density that the estimated transformation coincides with the true one, i.e. $\mathcal{L} = p(g|g) = \mathrm{Tr}[M(g)U_g\rho U_g^\dagger] = \mathrm{Tr}[\xi\rho]$. For a pure input state $\rho = |\psi\rangle\langle\psi|$, the maximization of the likelihood over all possible operators $\xi \geq 0$ satisfying the constraints (10) follows in a simple way by a repeated use of Schwartz inequality. In fact, the input state $|\psi\rangle$ can be written in the decomposition (4) as

$$|\psi\rangle = \bigoplus_{\mu\in\mathsf{S}} c_\mu\ |\Psi_\mu\rangle\!\rangle\ , \tag{11}$$

where each $|\Psi_\mu\rangle\!\rangle \in \mathcal{H}_\mu \otimes \mathcal{M}_\mu$ is a bipartite state. From Schwartz inequality, we have

$$\mathcal{L} \leq \sum_{\mu,\nu}\ \left|c_\mu^* c_\nu\ \langle\!\langle\Psi_\mu|\xi|\Psi_\nu\rangle\!\rangle\right| \leq \left(\sum_\mu |c_\mu| \sqrt{\langle\!\langle\Psi_\mu|\xi|\Psi_\mu\rangle\!\rangle}\right)^2 \tag{12}$$

At this point, we assume that each bipartite state $|\Psi_\mu\rangle\!\rangle$ is in the domain of the operator $D_\mu^{-1/2} \otimes \mathbb{1}_{\mathcal{M}_\mu}$. This assumption is not restrictive, since the domain of a self-adjoint operator is dense in the Hilbert space. In this way, it is possible to write $|\Psi_\mu\rangle\!\rangle = D_\mu^{1/2} D_\mu^{-1/2} \otimes \mathbb{1}_{\mathcal{M}_\mu}|\Psi_\mu\rangle\!\rangle$ and to exploit the Schmidt decomposition of the (non-normalized) vector $D_\mu^{-1/2} \otimes \mathbb{1}_{\mathcal{M}_\mu}\ |\Psi_\mu\rangle\!\rangle$. In other words, we can write

$$|\Psi_\mu\rangle\!\rangle = \sum_{m=1}^{r_\mu}\ \sqrt{\lambda_m^\mu}\ \sqrt{D_\mu} \otimes \mathbb{1}_{\mathcal{M}_\mu}|\tilde{\psi}_m^\mu\rangle|\tilde{\phi}_m^\mu\rangle\ , \tag{13}$$

where r_μ is the Schmidt rank, $\lambda^\mu_m \geq 0$ are Schmidt coefficients such that $\sum_{m=1}^{r_\mu} \lambda^\mu_m = \langle\langle \Psi_\mu | D_\mu^{-1} \otimes \mathbb{1}_{\mathcal{M}_\mu} | \Psi_\mu \rangle\rangle$, and $|\tilde{\psi}^\mu_m\rangle, |\tilde{\psi}^\mu_m\rangle$ are the elements of two orthonormal bases for $\mathcal{H}_\mu$ and $\mathcal{M}_\mu$, respectively. The form (13) is very convenient for optimization, in fact we can use again Schwartz inequality and obtain

$$\langle\langle \Psi_\mu | \xi | \Psi_\mu \rangle\rangle \leq \left(\sum_m \sqrt{\lambda^\mu_m \ \langle \tilde{\psi}^\mu_m | \langle \tilde{\phi}^\mu_m | \sqrt{D_\mu} \otimes \mathbb{1}_{\mathcal{M}_\mu} \xi \sqrt{D_\mu} \otimes \mathbb{1}_{\mathcal{M}_\mu} | \tilde{\psi}^\mu_m \rangle | \tilde{\phi}^\mu_m \rangle} \right)^2 \tag{14}$$

Finally, we have

$$\langle \tilde{\psi}^\mu_m | \langle \tilde{\phi}^\mu_m | \sqrt{D_\mu} \otimes \mathbb{1}_{\mathcal{M}_\mu} \xi \sqrt{D_\mu} \otimes \mathbb{1}_{\mathcal{M}_\mu} | \tilde{\psi}^\mu_m \rangle | \tilde{\phi}^\mu_m \rangle \leq \langle \tilde{\phi}^\mu_m | \ \mathrm{Tr}_{\mathcal{H}_\mu} [D_\mu \otimes \mathbb{1}_{\mathcal{M}_\mu} \xi] \ | \tilde{\phi}^\mu_m \rangle = 1 \ , \tag{15}$$

the last equality following from the normalization constraint (10). Therefore, the previous chain of inequalities proves the upper bound

$$\mathcal{L} \leq \left(\sum_{\mu \in \mathsf{S}} |c_\mu| \ \sum_{m=1}^{r_\mu} \sqrt{\lambda^\mu_m} \right)^2 = \mathcal{L}^{opt} \ , \tag{16}$$

that holds for any covariant POVM. On the other hand, it is immediate to check that the bound is achieved by the covariant POVM given by $\xi = |\eta\rangle\langle\eta|$ with

$$|\eta\rangle = \bigoplus_{\mu \in \mathsf{S}} e^{i \arg(c_\mu)} \ \sum_{m=1}^{r_\mu} D_\mu^{-1/2} \otimes \mathbb{1}_{\mathcal{M}_\mu} \ |\tilde{\psi}^\mu_m\rangle |\tilde{\phi}^\mu_m\rangle \ , \tag{17}$$

$\arg(z)$ denoting the argument of a complex number, i.e. $z = |z| e^{i \arg(z)}$. The normalization of such a POVM follows from Eq. (10), and one has

$$\int \mathrm{d}_L \, g U_g \xi U_g^\dagger = \bigoplus_\mu \mathbb{1}_{\mathcal{H}_\mu} \otimes \sum_{m=1}^{r_\mu} |\tilde{\phi}^\mu_m\rangle\langle\tilde{\phi}^\mu_m| \ , \tag{18}$$

namely, the POVM is complete in the subspace spanned by the orbit of $|\psi\rangle$, and can be trivially completed to the whole Hilbert space without affecting the probability distribution. Notice that, if the group $\mathbf{G}$ is unimodular, namely $D_\mu = \mathbb{1}_{\mathcal{H}_\mu}/d_\mu$ for some positive constant d_μ, then we correctly retrieve the results of Ref. [15] about maximum-likelihood measurements.

The case of $\{U_g\}$ being a direct sum of inequivalent irreps. The expression for the optimal covariant POVM can be further simplified in the case when all the multiplicity spaces $\mathcal{M}_\mu$ are one-dimensional, i.e. when the representation $\{U_g\}$ is a direct sum of inequivalent irreps. In this case, we can decompose the input state as $|\psi\rangle = \bigoplus_\mu \ c_\mu |\psi_\mu\rangle$ (as in Eq. (11), but without the need of introducing bipartite states), and now the decomposition (13) becomes trivial, namely

$$|\psi_\mu\rangle = \sqrt{\lambda_\mu} \ \sqrt{D_\mu} |\tilde{\psi}_\mu\rangle \ , \tag{19}$$

where $\lambda_\mu = \langle \psi_\mu | D_\mu^{-1} | \psi_\mu \rangle$ and

$$|\tilde{\psi}_\mu\rangle = \frac{D_\mu^{-1/2} |\psi_\mu\rangle}{|| D_\mu^{-1/2} |\psi_\mu\rangle ||} \ . \tag{20}$$

Therefore, Eq. (17) for the optimal POVM becomes

$$|\eta\rangle = \bigoplus_{\mu \in \mathsf{S}} \ e^{i \arg(c_\mu)} \ \frac{D_\mu^{-1} |\psi_\mu\rangle}{\sqrt{\langle \psi_\mu | D_\mu^{-1} | \psi_\mu \rangle}} \ = \ \bigoplus_{\mu \in \mathsf{S}} \ \frac{D_\mu^{-1} |\psi\rangle}{\sqrt{\langle \psi | D_\mu^{-1} | \psi \rangle}} \ , \tag{21}$$

and the corresponding optimal likelihood is given by

$$\mathcal{L}^{opt} = |\langle \eta | \psi \rangle|^2 = \left(\sum_{\mu \in \mathsf{S}} \sqrt{\langle \psi | D_\mu^{-1} | \psi \rangle} \right)^2 \ . \tag{22}$$

D. Remarks

Remark 1: Square-root measurements. A possible strategy to estimate an unknown quantum state, randomly drawn from a given family, is given by the so-called square-root measurements (SRM), firstly introduced by Hausladen and Wootters [22]. In the case of pure states with a group symmetry as in Eq. (1), there is an important connection between SRM and maximum-likelihood measurements. For example, if the group $\mathbf{G}$ is a discrete group of phase shifts, it has been proved in Refs. [25, 26, 27], that SRM minimize the probability of error in estimating the unknown state, and, equivalently, they maximize the likelihood, i.e. the probability of correct estimation. More generally, the optimality of the SRM in the maximum likelihood approach has been proved in Ref. [15] for a large class of groups, including all finite groups, all compact groups, and unimodular noncompact groups, such as the Weyl-Heisenberg group of displacements. However, as we will see in the following, the case of nonunimodular groups represents an exception to the fact that SRM are optimal for the maximum likelihood criterion in the presence of a physical symmetry. In fact, the SRM for the estimation of a group transformation acting on a fixed state ρ is given by the POVM $M_{sq}(g) = F^{-1/2}\ U_g \rho U_g^\dagger\ F^{-1/2}$, where

$$F = \int_{\mathbf{G}} \mathrm{d}_L\, g\ U_g \rho U_g^\dagger \tag{23}$$

(the POVM $M_{sq}(g)$ is obviously normalized with respect to the left-invariant measure $\mathrm{d}_L\, g$). The comparison with the maximum-likelihood measurements of the previous section is particularly simple in the case of group representations $\{U_g\}$ that are direct sum of inequivalent irreps. In fact, for a pure state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \bigoplus_\mu\ c_\mu |\psi_\mu\rangle$, the integral (23) is easily calculated by using Eq. (8), namely $F = \bigoplus_\mu |c_\mu|^2 \langle\psi_\mu|D_\mu|\psi_\mu\rangle\ \mathbb{1}_{\mathcal{H}_\mu}$. Notice here that the square-root measurement can be defined only if $|\psi_\mu\rangle$ is in the domain of $D_\mu^{1/2}$. Therefore, the square-root measurement is given by the covariant POVM $M_{sq}(g) = U_g|\eta_{sq}\rangle\langle\eta_{sq}|U_g^\dagger$, where

$$|\eta_{sq}\rangle = \bigoplus_\mu e^{i \arg(c_\mu)}\ \frac{|\psi_\mu\rangle}{\sqrt{\langle\psi_\mu|D_\mu|\psi_\mu\rangle}}\ . \tag{24}$$

This covariant POVM is different from the optimal one given in (21), and does not achieve the optimal value (22) for the likelihood. When $|\psi_\mu\rangle$ is in the domain of both $D_\mu^{1/2}$ and $D_\mu^{-1/2}$, we can compare the values of the likelihood as follows. One has

$$\mathcal{L}^{sq} = \left(\sum_\mu \frac{|c_\mu|}{\sqrt{\langle\psi_\mu|D_\mu|\psi_\mu\rangle}}\right)^2 \leq \left(\sum_\mu |c_\mu| \sqrt{\langle\psi_\mu|D_\mu^{-1}|\psi_\mu\rangle}\right)^2 = \mathcal{L}^{opt}\ , \tag{25}$$

where we used the inequality $\langle\psi_\mu|D_\mu^{-1}|\psi_\mu\rangle\ \langle\psi_\mu|D_\mu|\psi_\mu\rangle \geq 1$, from Schwartz inequality applied to the vectors $D_\mu^{1/2}|\psi_\mu\rangle$ and $D_\mu^{-1/2}|\psi_\mu\rangle$.

Remark 2: The true value and the most likely one. In the maximum likelihood approach, one optimizes the choice of the POVM in order to maximize the probability density that the estimated value of the parameters coincides with the true one. Intuitively, one could expect that the probability distribution $p(\hat{g}|g) = \mathrm{Tr}[M(\hat{g})U_g\rho U_g^\dagger]$ for the optimal POVM achieves its maximum at the value $\hat{g} = g$. Again, this is true for unimodular groups, but fails to hold for nonunimodular groups.

Proposition 2. *Let the group $\mathbf{G}$ be unimodular. If the covariant POVM $M(\hat{g})$ maximizes the likelihood for a given input state ρ, then the probability distribution $p(\hat{g}|g)$ of the estimate $\hat{g}$ on the state $U_g\rho U_g^\dagger$ achieves its maximum for $\hat{g} = g$.*

Proof. Suppose that the most likely value does not coincide with the true one. Then we can rigidly shift the whole probability distribution with a post-processing operation that brings the most likely value to the true one. In fact, if the maximum of $p(\hat{g}|g)$ occurs at $\hat{g} = gh$, we can always replace $M(\hat{g})$ with a new covariant POVM $M'(\hat{g}) = U_{\hat{g}}\ \xi'\ U_{\hat{g}}^\dagger$, where

$$\xi' = U_h \xi U_h^\dagger\ . \tag{26}$$

The normalization of the new POVM follows from the fact that for unimodular groups the DMC operators are trivially proportional to the identity, and therefore the operator ξ' satisfy the normalization constraints (10) as well. Moreover, the probability distribution $p'(\hat{g}|g)$ associated with $M'(\hat{g})$ enjoys the property $p'(\hat{g}|g) = p(\hat{g}h|g)$, whence it achieves

the maximum in $\hat{g} = g$. In this way, the likelihood of $M'(\hat{g})$ would be higher than the likelihood of the POVM $M(\hat{g})$. But this cannot happen since $M(\hat{g})$ is the optimal maximum-likelihood POVM. Therefore $p(\hat{g}|g)$ must be maximum in $\hat{g} = g$. ■

For nonunimodular groups the previous argument does not apply, since the POVM given by (26) is no longer normalized. In fact, the operator ξ' does not satisfy the normalization constraints (10), since the DMC operators do not commute with the unitaries U_h. In other words, we are not allowed to bring the most likely value to coincide with the true one by rigidly shifting the whole probability distribution. As we will see in the explicit example of the estimation of real squeezing and displacement, this situation can indeed happen. In order to reduce the discrepancy between the true value and the most likely one, a suitable choice of the input states is needed. For example, in the simple case of $\{U_g\}$ being a direct sum of inequivalent irreps, if the projection of the input state onto the irreducible subspaces are eigenvectors of the DMC operators, then the most likely value coincides with the true one. In fact, for any input state $|\psi\rangle = \bigoplus_\mu \ c_\mu|\psi_\mu\rangle$, using Schwarz inequality, we have

$$p^{opt}(\hat{g}|g) = \left|\sum_\mu |c_\mu| \frac{\langle\psi_\mu|U_{g^{-1}\hat{g}}D_\mu^{-1}|\psi_\mu\rangle}{\sqrt{\langle\psi_\mu|D_\mu^{-1}|\psi_\mu\rangle}}\right|^2 \leq \left(\sum_\mu |c_\mu|\sqrt{\frac{\langle\psi_\mu|D_\mu^{-2}|\psi_\mu\rangle}{\langle\psi_\mu|D_\mu^{-1}|\psi_\mu\rangle}}\right)^2 , \tag{27}$$

and if each $|\psi_\mu\rangle$ is eigenvector of D_μ, then the last expression is equal to $p(g|g)$, then the true value is the most likely one.

III. OPTIMAL ESTIMATION OF REAL SQUEEZING AND DISPLACEMENT

A. Translation and dilation

In the following we will apply the general framework of Section II to the case of joint estimation of real squeezing and displacement of a single-mode radiation field with bosonic operators a and $a^\dagger$ with $[a, a^\dagger] = 1$. Given the wavefunction of a pure state $|\psi\rangle$ in the X-representation $\psi(x) = \langle x|\psi\rangle$, where $|x\rangle$ denotes the Dirac-normalized eigenstate of the quadrature operator $X = (a + a^\dagger)/2$, the affine transformation on the real line given by $x \to e^r x + x'$ is represented by the unitary transformation

$$\psi(x) \to e^{-r/2} \ \psi(e^{-r}(x - x')) , \qquad x', r \in \mathbb{R} . \tag{28}$$

This transformation corresponds to the action of the unitary operator $U_{x',r} = D(x')S(r)$ on the ket $|\psi\rangle$, where

$$\begin{aligned} D(x) &= \exp\left[x(a^\dagger - a)\right] , \\ S(r) &= \exp\left[\frac{r}{2}(a^{\dagger 2} - a^2)\right] , \end{aligned} \tag{29}$$

represent the displacement and the squeezing operator with real argument, respectively. In other words, the operators

$$\{U_{x,r} = D(x)S(r) \mid x, r \in \mathbb{R}\} , \tag{30}$$

provide a unitary representation of the affine group in the Hilbert space of wavefunctions. The affine group is nonunimodular, and in the above parametrization the left- and right-invariant measures are given by $\mathrm{d}_L\, g = e^{-r}\,\mathrm{d}\, r\,\mathrm{d}\, x$, and $\mathrm{d}_R\, g = \mathrm{d}\, r\,\mathrm{d}\, x$, respectively.

B. The maximum likelihood POVM

In order to exploit the general results of Sec. II, we need to know the Clebsch-Gordan decomposition of the representation $\{U_{x,r}\}$, the irreducible subspaces, and the DMC operators. All these informations are given in the following, while the proof is presented in the Appendix.

The Clebsch-Gordan series of the representation $\{U_{x,r}\}$ consists on two irreps, that we indicate with the symbols $+$ and $-$. Accordingly, the Hilbert space splits into two irreducible subspaces, i.e.

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- . \tag{31}$$

Comparing this decomposition with the general case (4), we see that the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are the representation spaces, while the multiplicity spaces $\mathcal{M}_+$ and $\mathcal{M}_-$ are trivially one-dimensional. The representation spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ can be easily characterized in terms of the quadrature $Y = \frac{a-a^\dagger}{2i}$. In fact, writing the wavefunctions in the Y-representation as $\psi(y) = \langle y|\psi\rangle$, where $|y\rangle$ are the Dirac-normalized eigenvectors of Y, we have $\mathcal{H}_+ = \{|\psi\rangle \mid \psi(y) = 0 \quad \forall y < 0\}$ and $\mathcal{H}_- = \{|\psi\rangle \mid \psi(y) = 0 \quad \forall y > 0\}$. Therefore, the projection operators onto $\mathcal{H}_+$ and $\mathcal{H}_-$ can be written respectively as $\mathbb{1}_+ = \theta(Y)$ and $\mathbb{1}_- = \theta(-Y)$, where $\theta(x)$ is the customary step-function [$\theta(x) = 1$ for $x \geq 0$, $\theta(x) = 0$ for $x < 0$]. Moreover, the DMC operators are

$$D_\pm = \pi \, \frac{\theta(\pm Y)}{|Y|} \, . \tag{32}$$

With these tools we are now able to provide the optimal covariant measurement for the joint estimation of real squeezing and displacement on a given state of the radiation field. Let us denote by $|\psi\rangle$ the input state that undergoes to unknown squeezing and displacement transformations. Decomposing the input state on the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ as $|\psi\rangle = c_+|\psi_+\rangle + c_-|\psi_-\rangle$, we can exploit Eq. (21) and write explicitly the optimal POVM as $M(x,r) = U_{x,r}|\eta\rangle\langle\eta|U^\dagger_{x,r}$ where

$$|\eta\rangle = \frac{|Y|\theta(Y)|\psi\rangle}{\sqrt{\pi \, \langle\psi||Y|\theta(Y)|\psi\rangle}} + \frac{|Y|\theta(-Y)|\psi\rangle}{\sqrt{\pi \, \langle\psi||Y|\theta(-Y)|\psi\rangle}} \, . \tag{33}$$

The optimal likelihood is then

$$\mathcal{L}^{opt} = \frac{1}{\pi} \, \left(\sqrt{\langle\psi| \; |Y|\theta(Y) \; |\psi\rangle} + \sqrt{\langle\psi| \; |Y|\theta(-Y) \; |\psi\rangle}\right)^2 \, , \tag{34}$$

according to the general expression of Eq. (22). As already mentioned, the expression of the likelihood provides some insight about the states that are most sensitive in detecting an unknown combination of real squeezing and displacement. Essentially, one can improve the likelihood by increasing the expectation value of $|Y|$, the modulus of the quadrature Y. In addiction, the use of wavefunctions that in the Y-representation are non-zero both in the positive half-line and in the negative half-line allows to exploit the interference of the components $|\psi_+\rangle$ and $|\psi_-\rangle$ to enhance the value of the likelihood.

C. Real squeezing and displacement on a coherent state.

Using Eq. (33) for the optimal POVM, we can obtain the probability distribution of the estimated squeezing and displacement parameter for a given input state. In particular, for a coherent input state $|\alpha\rangle$ the sensitivity of the measurement can be significantly improved by increasing the imaginary part of α, this corresponding to taking coherent states with a high expectation value of $|Y|$. The probability distribution for the joint estimation of squeezing and displacement on the vacuum state and on a coherent state with $\alpha = 10i$ has been reported in Figs. 1 and 2, respectively. Comparing the two figures, a remarkable improvement in the precision of the measurement can be observed as an enhancement of the likelihood, along with a narrowing of the probability distribution. Moreover, we can observe that for the vacuum state the maximum of the probability density is not achieved by the true value (which is given by to $x = r = 0$). The discrepancy between the most likely value and the true one, due to the fact that the affine group is nonunimodular, essentially disappears by increasing the expectation value of $|Y|$. As we can see from Fig. 2, for $\alpha = 10i$ the probability distribution is approximately a Gaussian centered around the true value $x = r = 0$.

Now it is interesting to focus on the asymptotic behavior of the probability distribution for a coherent state $|ia\rangle$ with $a \in \mathbb{R}$ going to infinity. In the asymptotic regime, the probability distribution

$$p_a(x,r) \, \mathrm{d}\,x \, \mathrm{d}\,r e^{-r} = |\langle\eta| \; U_{x,r} \; |ia\rangle|^2 \, \mathrm{d}\,x \, \mathrm{d}\,r e^{-r} \, , \tag{35}$$

given by optimal vector $|\eta\rangle$ in Eq. (33), can be further simplified. In fact, the wavefunction of the coherent state can be written as

$$\langle y|ia\rangle = \left(\frac{2}{\pi}\right)^{1/4} \; e^{-(y-a)^2} \quad , \tag{36}$$

and for $a \gg 1$ it is essentially confined in the positive part of the y-axis. Hence, in the expression (33) we can asymptotically neglect the component in the subspace $\mathcal{H}_-$ and drop the modulus from $|Y|$. In this way, the probability distribution (35) can be approximated as

$$p_a(x,r) \, \mathrm{d}\,x \, \mathrm{d}\,r e^{-r} \approx \frac{1}{\pi} \, \frac{|\langle ia| \; U_{x,r} Y \; |ia\rangle|^2}{\langle ia| \; Y \; |ia\rangle} \; \mathrm{d}\,x \, \mathrm{d}\,r e^{-r} \, . \tag{37}$$

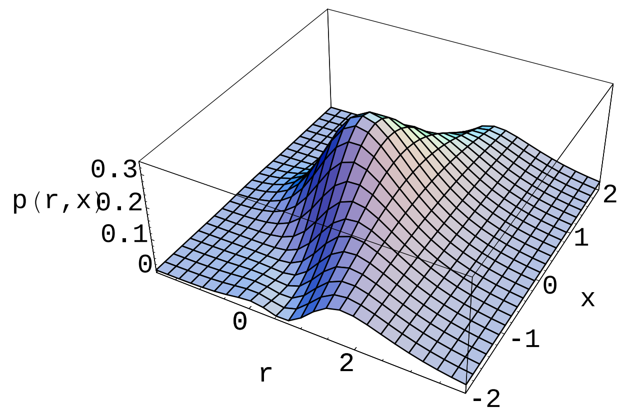

FIG. 1: Optimal probability distribution for the joint estimation of the squeezing parameter and real displacement for the vacuum state.

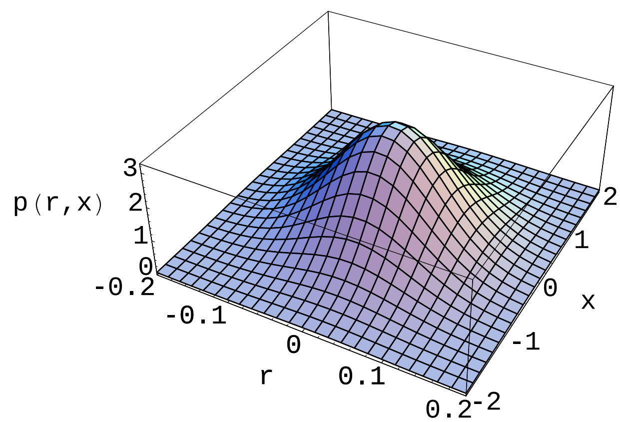

FIG. 2: Optimal probability distribution for the joint estimation of the squeezing parameter and real displacement for a coherent state with amplitude $\alpha = 10i$.

Neglecting the higher order terms, we thus obtain the Gaussian distribution

$$p_a(x,r)\,\mathrm{d}\,x\,\mathrm{d}\,re^{-r} = \frac{a}{\pi}\;e^{-a^2r^2}\;e^{-x^2}\;\;\mathrm{d}\,x\,\mathrm{d}\,r\;. \tag{38}$$

Notice that, asymptotically the most likely values of the unknown parameters x and r are the true ones $x = r = 0$, and, in addiction, also the mean values of x and r coincide with the true one, namely the estimation is unbiased.

From the asymptotic expression (38) can see that the uncertainty in the estimation of the squeezing parameter r goes to zero with the number of photons $\bar{n} = a^2$, namely the r.m.s error is $\Delta r = 1/\sqrt{2\bar{n}}$, while the uncertainty in the estimation of the displacement x remains fixed, with the value $\Delta x = 1/\sqrt{2}$. It is interesting to compare the precision achieved by the joint estimation with the precision that could be achieved if the parameters x and r were measured separately. First, it is known that the optimal estimation of real displacement is given by the observable $X = (a + a^\dagger)/2$, and the corresponding uncertainty in a coherent state is given by $\Delta x^{opt} = 1/2$. The optimal estimation of real squeezing has been recently derived in [29], and we report here the asymptotic distribution for an

excited coherent state:

$$p(r) = \sqrt{\frac{2|\alpha|^2}{\pi}}\; e^{-2|\alpha|^2 r^2} \; . \tag{39}$$

Accordingly, the r.m.s. error for the optimal estimation of squeezing in a coherent state is $\Delta r^{opt} = 1/(2\sqrt{\bar{n}})$. It is remarkable to note the relation

$$\begin{cases} \Delta x &= \sqrt{2}\, \Delta x^{opt} \\ \Delta r &= \sqrt{2}\, \Delta r^{opt} \end{cases} \tag{40}$$

between the uncertainties in the joint measurement and the ones in the separate measurements of squeezing and displacement. In particular, the product of the uncertainties in the joint estimation is twice the product of uncertainties in the optimal separate measurements:

$$\Delta x \Delta r = 2 \Delta x^{opt}\; \Delta r^{opt} \; . \tag{41}$$

Surprisingly, this is exactly the same relation as the one occurring in the optimal joint measurement of two conjugated quadratures X and Y that can be achieved by heterodyne detection [30].

D. Joint estimation for a displaced squeezed state

In the previous section we analyzed the optimal estimation of squeezing and displacement for an excited coherent state. In that case, while the error in the estimation of r goes to zero with the number of photons, the error in estimating x remains fixed. However, it is possible to choose the input state in such a way that both variances vanish in the asymptotic limit. To this purpose, we consider here displaced squeezed states $|ia,z\rangle = D(ia)S(z)|0\rangle$ with $a, z \in \mathbb{R}$. Such states have the wavefunction

$$\langle y|ia,z\rangle = \left(\frac{2e^{2z}}{\pi}\right)^{1/4} \; e^{-(y-a)^2 e^{2z}} \; , \tag{42}$$

namely a Gaussian centered around the mean value a, with standard deviation $\sigma = 1/(\sqrt{2}e^{z})$. Clearly, if the conditions $a \gg 1$ and $a \gg \sigma$ are simultaneously satisfied, such a Gaussian lies almost completely in the positive half-line. Therefore, in the asymptotic limit $a \to +\infty$, $a \gg e^{-z}$, the optimal probability distribution can be approximated as

$$p_{a,z}(x,r) \approx \frac{1}{\pi}\; \frac{|\langle ia,z|\; U_{x,r} Y \; |ia,z\rangle|^2}{\langle ia,z|\; Y\; |ia,z\rangle} \; , \tag{43}$$

as in the case of coherent states. By calculating the expectation values and keeping the leading order terms, we then obtain the asymptotic distribution

$$p_{a,z}(x,r)\, \mathrm{d}\, x\, \mathrm{d}\, r e^{-r} = \frac{a}{\pi} e^{-(ae^{z} r)^2} e^{-(x e^{-z})^2}\, \mathrm{d}\, x\, \mathrm{d}\, r \; . \tag{44}$$

Again, in the asymptotic limit the most likely values in the probability distribution coincide with the true ones, and, moreover, the estimation is unbiased.

The r.m.s. error in the estimation of squeezing and displacement are now given by $\Delta r = 1/(\sqrt{2}ae^{z})$ and $\Delta x = 1/(\sqrt{2}e^{-z})$, respectively. In order to have both errors vanishing, one needs simultaneously $ae^{z} \gg 1$ and $e^{-z} \gg 1$. For example, we can have an isotropic distribution $\Delta r = \Delta x$ with the choice $a = e^{-2z}$. In the isotropic case, we notice that only a small fraction of order $\sqrt{\bar{n}}$ of the total number of photons $\bar{n} = a^2 + \sinh^2 z$ comes from squeezing. Since one has $\Delta x = \Delta r = 1/(\sqrt{2}e^{-z}) \approx 1/(\sqrt{2}\bar{n}^{1/4})$, the convergence to the asymptotic regime is quite slow: the uncertainty goes to zero with order $1/\bar{n}^{1/4}$. As an example in the asymptotic regime, in Fig. 3 we plot the exact probability distribution for $\bar{n} = 4000$.

Also for displaced squeezed states one can compare the accuracy of the joint estimation with that of independent measurements of squeezing and displacement. For squeezed states of the form (42) the uncertainty in the measurement of the observable X is $\Delta x^{opt} = 1/(2e^{-z})$. On the other hand, the optimal estimation of squeezing is obtained asymptotically by the observable $\ln(|Y|/a)$, and its uncertainty is $\Delta r^{opt} = 1/(2ae^{z})$ (according to the results of [29]). Again, a factor $\sqrt{2}$ relates the standard deviations of the marginals in the joint estimation with those of the optimal separate measurements. Again, we find the relation (41) between the product of uncertainties.

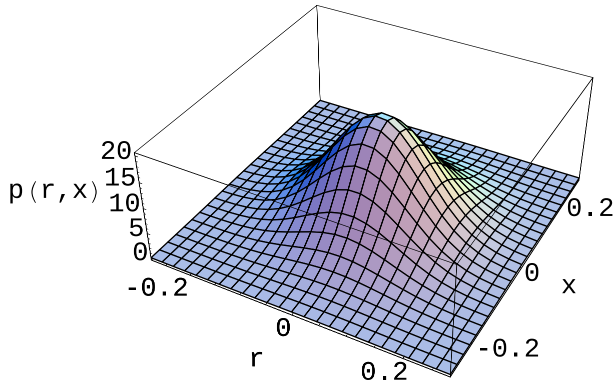

FIG. 3: Optimal probability distribution for the joint estimation on a displaced squeezed states with total number of photons $\bar{n} = 4000$ (with signal photons $\bar{n} - \sqrt{\bar{n}}$, and squeezing photons $\sqrt{\bar{n}}$).

We then see that the estimation of real squeezing and displacement on input states of the form (42) corresponds in the asymptotic regime to the joint measurement of the observables X and $(\ln |Y|/a)$, whose commutator is $[X, \ln(|Y|/a)] = i/(2Y)$. We thus have the Heisenberg-Robertson inequality

$$\Delta X \Delta \ln(|Y|/a) \geq \left| \langle \frac{1}{4Y} \rangle \right| , \tag{45}$$

where $\langle \ \rangle$ denotes the expectation value. From this point of view, the displaced squeezed states (42) are characterized asymptotically as minimum uncertainty states, since they saturate the inequality, and the product of uncertainties in the maximum likelihood estimation is exactly twice the Heisenberg limit. To the best of our knowledge, the present case is the first example of joint measurement of two harmonic-oscillator noncommuting observables whose commutator is not a c-number.

E. Joint estimation of squeezing, displacement, and reflection

The results presented in the previous sections can be easily extended to include the estimation of the reflection on the real line that is realized by the parity operator $P = (-1)^{a^\dagger a}$. In this case we are interested in estimating the three parameters in the transformation

$$|\psi\rangle \longrightarrow P^\epsilon D(x) S(r) |\psi\rangle , \tag{46}$$

where ϵ can assume the values 0 or 1. The representation $\{P^\epsilon D(x) S(r)\}$ is now irreducible in $\mathcal{H}$, and the associated DMC operator is $D = \pi/|Y|$. For a given input state $|\psi\rangle$, the optimal POVM can be written as $M(\epsilon, x, r) = P^\epsilon D(x) S(r) |\eta\rangle\langle\eta| S(r)^\dagger D(x)^\dagger P^\epsilon$, where

$$|\eta\rangle = \frac{|Y||\psi\rangle}{\sqrt{\pi \ \langle\psi| \ |Y| \ |\psi\rangle}} , \tag{47}$$

according to Eq. (21). Clearly, also in this case it is possible to enhance the sensitivity of detection by increasing the average value of the modulus $|Y|$, using coherent states or displaced squeezed states.

F. Perfect detection of squeezing and displacement: a two-mode setup

A fundamental mechanism leading to the optimal estimation of group transformations is the use of equivalent representations of the group, via the technique of entanglement between representation spaces and multiplicity [15, 16].

Such a strategy strongly improves the quality of estimation for unimodular groups [12, 14, 15], and in the case of compact groups is a way to obtain the best estimation of a unitary transformation[16]. Equivalent representations of the affine group can be obtained by entangling the radiation with a second reference mode, which is not affected by the unknown squeezing and displacement. This corresponds to considering the two-mode Hilbert space $\mathcal{H} \otimes \mathcal{H}$ and the representation $\{U_{x,r} \otimes \mathbb{1}\}$ of the affine group, namely the reference mode plays the role of an infinite dimensional multiplicity space. In this case, the optimal estimation can be obtained with the POVM specified by the general formula (17). A remarkable feature of the two-mode setup is that it is possible to have an *orthogonal* POVM for the estimation, namely there exists an ordinary observable on the extended Hilbert space, associated with the joint measurement of squeezing and displacement. In fact, by defining the bipartite vectors

$$|\Phi_\pm\rangle\!\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\, y\ \sqrt{|y|}\ |y\rangle| \pm y\rangle\ , \tag{48}$$

we can construct the orthogonal POVM

$$M(g) = U_{x,r} \otimes \mathbb{1} \left(\sum_{s=\pm} |\Phi_s\rangle\!\rangle\langle\!\langle\Phi_s| \right) U_{x,r}^\dagger \otimes \mathbb{1}\ . \tag{49}$$

The normalization follows straightforwardly from Eq. (9). Moreover, we have the orthogonality relation

$$\begin{aligned} \langle\!\langle\Phi_s|\ U_{x,r} \otimes \mathbb{1}\ |\Phi_{s'}\rangle\!\rangle &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\, y\ |y|\ \langle y|U_{x,r}|sy/s'\rangle \\ = \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\, y\ |y|\ e^{-r/2} e^{-2ixye^{-r}}\ \delta(y - ye^{-r})\, \delta_{s,s'} &= \delta(x)\delta(r)\delta_{s,s'}\ , \end{aligned} \tag{50}$$

where we used Eq. (52) of the Appendix and the identity $\delta(y - ye^{-r}) = \frac{1}{|y|}\delta(r)$. The vectors in Eq. (48) are not normalizable. However, similarly to the case of the heterodyne operator, where physical states can arbitrarily well approximate the unnormalizable eigenstates [28], here one can consider, e.g., states of the form

$$|\Phi(\lambda)_\pm\rangle\!\rangle = N_\lambda\ \lambda^{a^\dagger a + b^\dagger b}\ |\Phi_\pm\rangle\!\rangle\ , \tag{51}$$

where $0 < \lambda < 1$, N_λ is a normalization constant, and $b^\dagger b$ is the photon number operator for the auxiliary mode b. The states $|\Phi(\lambda)_\pm\rangle\!\rangle$ approaches the optimal vectors (48) as $\lambda \to 1$, with a correspondent increasing to infinity of the average energy of the radiation field.

IV. CONCLUSIONS

In this paper we presented the joint estimation of real squeezing and displacement of the radiation field from the general point of view of group parameter estimation. The combination of squeezing and displacement provides a representation of the affine group "$ax + b$", which is the paradigmatic example of a nonunimodular group. To deal with the concrete example of squeezing and displacement, we derived in the maximum likelihood approach the optimal estimation of a group transformation in the case of nonunimodular groups, providing explicitly the optimal POVM for a given input state. In this analysis, some remarkable features of estimation showed up. Firstly, while for unimodular groups the maximum likelihood measurements coincide with the usual square-root measurements, for nonunimodular groups the SRM are no longer optimal, namely they do not maximize the probability density of detecting the correct value. Moreover, for nonunimodular groups one can optimize the estimation strategy in the maximum likelihood approach, but even for the optimal POVM the true value is not the one which is most likely to be detected. To reduce the discrepancy between the true value and the most likely one a suitable choice of the input states is required. Both these features are in general unavoidable, and their origin tracks back to the presence of a positive unbounded operator—the Duflo-Moore-Carey operator—in the orthogonality relations for nonunimodular groups. In the problem of joint estimating real squeezing and displacement, all the above effects occur. In particular, for coherent input states and displaced squeezed states we observed how an increase in the expectation value of $|Y|$ gives rise to an improvement in the quality of estimation, along with a reduction of the discrepancy between the true value and the most likely one. In the mentioned cases the probability distributions for joint estimation become asymptotically Gaussian, and the r.m.s. errors Δx and Δr can be easily calculated. Remarkably, the product of uncertainties in the joint estimation is exactly twice the product of uncertainties for the optimal separate measurements of squeezing and displacement, in the same way as in the joint measurement of two conjugated quadratures. Finally, the use of entanglement with an

additional mode of the radiation field allows one to perform a von Neumann measurement for the joint estimation of squeezing and displacement, in terms of an ordinary observable with continuous spectrum. The ideal input states for detecting an unknown affine transformation can then be approximated by normalizable states.

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V. APPENDIX

Here we derive the group theoretical structure of the representation $\{U_{x,r} = D(x)S(r)\}$ of the affine group, by explicitly calculating the expression for the group average of an operator A over the left-invariant measure $\mathrm{d}_L\, g = \mathrm{d}\,x\,\mathrm{d}\,re^{-r}$.

Proposition 3. *The Clebsch-Gordan series of $\{U_{x,r}\}$ contains two irreducible irreps, $+$ and $-$. Accordingly, the Hilbert space can be decomposed as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and the projections onto the irreducible subspaces are $1\!\!1_+ = \theta(Y)$ and $1\!\!1_- = \theta(-Y)$, respectively. The DMC operators are given by $D_+ = \pi\theta(Y)/|Y|$ and $D_- = \pi\theta(-Y)/|Y|$.*

Proof. Using twice the resolution of the identity in terms of the eigenstates $|y\rangle$ of the quadrature operator Y, and the relation

$$U_{x,r}|y\rangle = D(x)S(r)|y\rangle = D(x)e^{-r/2}|e^{-r}y\rangle = e^{-r/2}e^{-2ixye^{-r}}|e^{-r}y\rangle \;, \tag{52}$$

we can calculate the group average as follows

$$\begin{aligned}
\langle A\rangle &= \int_{-\infty}^{\infty}\mathrm{d}\,re^{-r}\int_{-\infty}^{\infty}\mathrm{d}\,x\,U_{x,r}AU^\dagger_{x,r}\\
&= \int_{-\infty}^{\infty}\mathrm{d}\,re^{-r}\int_{-\infty}^{\infty}\mathrm{d}\,x\int_{-\infty}^{\infty}\mathrm{d}\,y\int_{-\infty}^{\infty}\mathrm{d}\,y'\,\langle y|A|y'\rangle\, e^{-2ix(y-y')e^{-r}}\, e^{-r}|e^{-r}y\rangle\langle e^{-r}y'|\\
&= \pi\int_{-\infty}^{\infty}\mathrm{d}\,r\int_{-\infty}^{\infty}\mathrm{d}\,y\,\langle y|A|y\rangle\, e^{-r}|e^{-r}y\rangle\langle e^{-r}y|\\
&= \pi\int_{-\infty}^{\infty}\mathrm{d}\,y\int_{0}^{\mathrm{sgn}(y)\infty}\mathrm{d}\,\tilde r\,\frac{1}{y}\,\langle y|A|y\rangle\,|\tilde r\rangle\langle\tilde r|\\
&= \pi\int_0^{\infty}\mathrm{d}\,\tilde r\,|\tilde r\rangle\langle\tilde r|\ \mathrm{Tr}[A\theta(Y)/Y] - \pi\int_{-\infty}^{0}\mathrm{d}\,\tilde r\,|\tilde r\rangle\langle\tilde r|\ \mathrm{Tr}[A\theta(-Y)/Y]\\
&= \theta(Y)\ \ \mathrm{Tr}[A\ \pi\theta(Y)/|Y|] + \theta(-Y)\ \ \mathrm{Tr}[A\ \pi\theta(-Y)/|Y|]\;.
\end{aligned} \tag{53}$$

The thesis follows by comparing the last equation with the general formula (8) for the group average.

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